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ECE 204 *Numerical methods*

Periodic functions and Fourier series

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Introduction

- In this topic, we will
  - Describe an inner product of two functions
  - Define orthogonality of functions
  - Show that we can project one function onto another
  - Discuss projections onto a set of mutually orthogonal functions
  - Introduce complex exponential functions
  - Show that these functions are mutually orthogonal
  - Approximate periodic functions with these complex exponential functions

An inner product for functions

- Suppose we have an interval $[-\frac{1}{2}T, \frac{1}{2}T]$,
  and given two piecewise continuous periodic functions
  f and g with period T, we may define

$$\langle f, g \rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* g(t) \, \mathrm{d}t$$

- Contrast this with $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^{n} u_k^* v_k$

An inner product for functions

- This has all the properties of the inner product of finite-dimensional vectors

$$\langle f, f\rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* f(t)\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2\,\mathrm{d}t \geq 0$$

$$\langle f, g\rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* g(t)\,\mathrm{d}t = \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)^* f(t)\,\mathrm{d}t\right)^* = \langle g, f\rangle^*$$

$$\langle f, \alpha g\rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* \alpha g(t)\,\mathrm{d}t = \alpha\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* g(t)\,\mathrm{d}t = \alpha\langle f, g\rangle$$

$$\langle f, g+h\rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* \left(g(t)+h(t)\right)\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[f(t)^* g(t) + f(t)^* h(t)\right]\mathrm{d}t$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* g(t)\,\mathrm{d}t + \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* h(t)\,\mathrm{d}t = \langle f, g\rangle + \langle f, h\rangle$$

An inner product for functions

- Two functions are orthogonal if $\langle f, g \rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* g(t)\,\mathrm{d}t = 0$
- We can define the 2-norm of a function:

$$\| f \|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 \,\mathrm{d}t}$$

- We can also define the projection of one function onto another

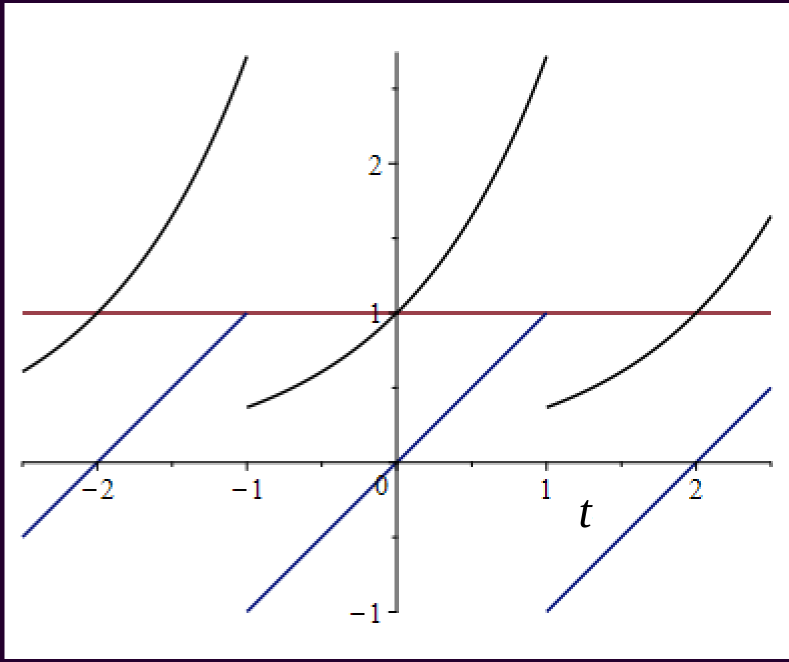
$$\mathrm{proj}_f(g) = \frac{\langle f, g \rangle}{\langle f, f \rangle} f = \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* g(t)\,\mathrm{d}t}{\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)^* f(t)\,\mathrm{d}t} f$$

Two orthogonal functions and a projection

- For example, the constant function f_0 together with the saw-tooth function f_1 with period $T = 2$ form an orthogonal set:

$$\langle f_0, f_1 \rangle = \int_{-1}^{1} (1 \cdot t)\,\mathrm{d}t = \frac{1}{2}t^2\Big|_{-1}^{1} = 0$$

  – If $g(t) = e^t$ on $[-1, 1]$ is periodically extended



Two orthogonal functions and a projection

- Doing the calculations:

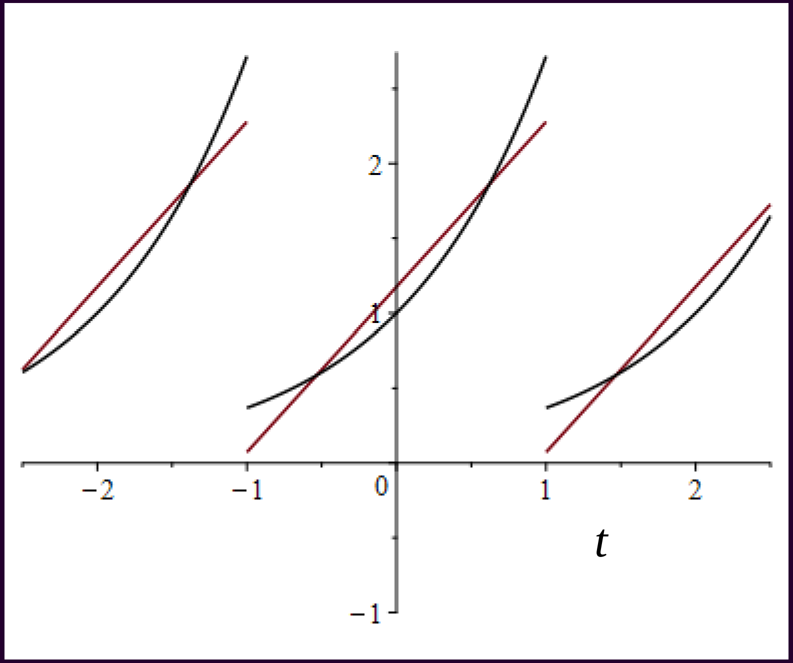
$$\operatorname{proj}_{f_0}(g) = \frac{\langle f_0, g\rangle}{\langle f_0, f_0\rangle} f_0 = \frac{\int_{-1}^{1} 1\cdot e^t \,\mathrm{d}t}{\int_{-1}^{1} 1^2 \,\mathrm{d}t} f_0 \qquad a_0 = \frac{\langle f_0, g\rangle}{\langle f_0, f_0\rangle} = \frac{e - e^{-1}}{2}$$

$$\operatorname{proj}_{f_1}(g) = \frac{\langle f_1, g\rangle}{\langle f_1, f_1\rangle} f_1 = \frac{\int_{-1}^{1} t e^t \,\mathrm{d}t}{\int_{-1}^{1} t^2 \,\mathrm{d}t} f_1 \qquad a_1 = \frac{\langle f_1, g\rangle}{\langle f_1, f_1\rangle} = \frac{2e^{-1}}{\frac{2}{3}} = 3e^{-1}$$

Two orthogonal functions and a projection

- Thus, the best approximation of the periodically extended function e^t as a linear combination is the periodic extension of

$$\frac{e-e^{-1}}{2}\cdot 1+3e^{-1}t$$



Orthogonal complex exponentials

- Next, consider the complex-valued functions

$$u_n(t) = e^{jn\frac{2\pi}{T}t}$$

where n is any integer

$$\langle u_m, u_n \rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(e^{jm\frac{2\pi}{T}t}\right)^* \left(e^{jn\frac{2\pi}{T}t}\right) \mathrm{d}t$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-jm\frac{2\pi}{T}t} e^{jn\frac{2\pi}{T}t} \, \mathrm{d}t \qquad = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(n-m)\frac{2\pi}{T}t} \, \mathrm{d}t$$

Orthogonal complex exponentials

- Now, if $m = n$, we have:

$$\langle u_m, u_m \rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(m-m)\frac{2\pi}{T}t}\,\mathrm{d}t$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j0\frac{2\pi}{T}t}\,\mathrm{d}t$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} 1\,\mathrm{d}t = T$$

Orthogonal complex exponentials

- Otherwise, if $m \neq n$, we have:

$$\langle u_m, u_n \rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(n-m)\frac{2\pi}{T}t}\,\mathrm{d}t = \frac{1}{j(n-m)\frac{2\pi}{T}} e^{j(n-m)\frac{2\pi}{T}t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$\int_a^b e^{st}\,\mathrm{d}t = \frac{1}{s}e^{st}\Big|_a^b$$

$$= \frac{1}{j(n-m)\frac{2\pi}{T}}\left(e^{j(n-m)\frac{2\pi}{T}\frac{T}{2}} - e^{j(n-m)\frac{2\pi}{T}\left(-\frac{T}{2}\right)}\right)$$

$$= \frac{1}{j(n-m)\frac{2\pi}{T}}\left(e^{j(n-m)\pi} - e^{-j(n-m)\pi}\right)$$

$$= \frac{1}{j(n-m)\frac{2\pi}{T}}\left(e^{j(n-m)\pi} - \left(e^{j(n-m)\pi}\right)^*\right)$$

Orthogonal complex exponentials

- Recall that $z - z^* = 2j\,\Im\mathrm{m}(z)$

$$\langle u_m, u_n \rangle = \frac{1}{j(n-m)\frac{2\pi}{T}}\left(e^{j(n-m)\pi} - \left(e^{j(n-m)\pi}\right)^*\right)$$

$$e^{js} = \cos(s) + j\sin(s)$$

$$= \frac{1}{j(n-m)\frac{2\pi}{T}} 2j\sin((n-m)\pi)$$

$$= \frac{T}{\pi(n-m)}\sin((n-m)\pi) \quad = 0$$

Orthogonal complex exponentials

- Thus, if $u_n(t) = e^{jn\frac{2\pi}{T}t}$, we have

$$\langle u_m, u_n \rangle = \begin{cases} T & m = n \\ 0 & m \neq 0 \end{cases}$$

$$\langle u_m, u_n \rangle = T \operatorname{sinc}(n - m)$$

$$\operatorname{sinc}(x) \overset{\text{def}}{=} \frac{\sin(\pi x)}{\pi x}$$

Orthogonal complex exponentials

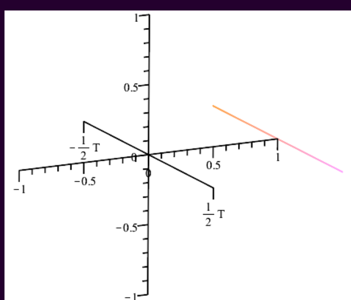
- Question: what is the span of all functions

$$\ldots, u_{-2}(t), u_{-1}(t), u_0(t), u_1(t), u_2(t), \ldots\ ?$$

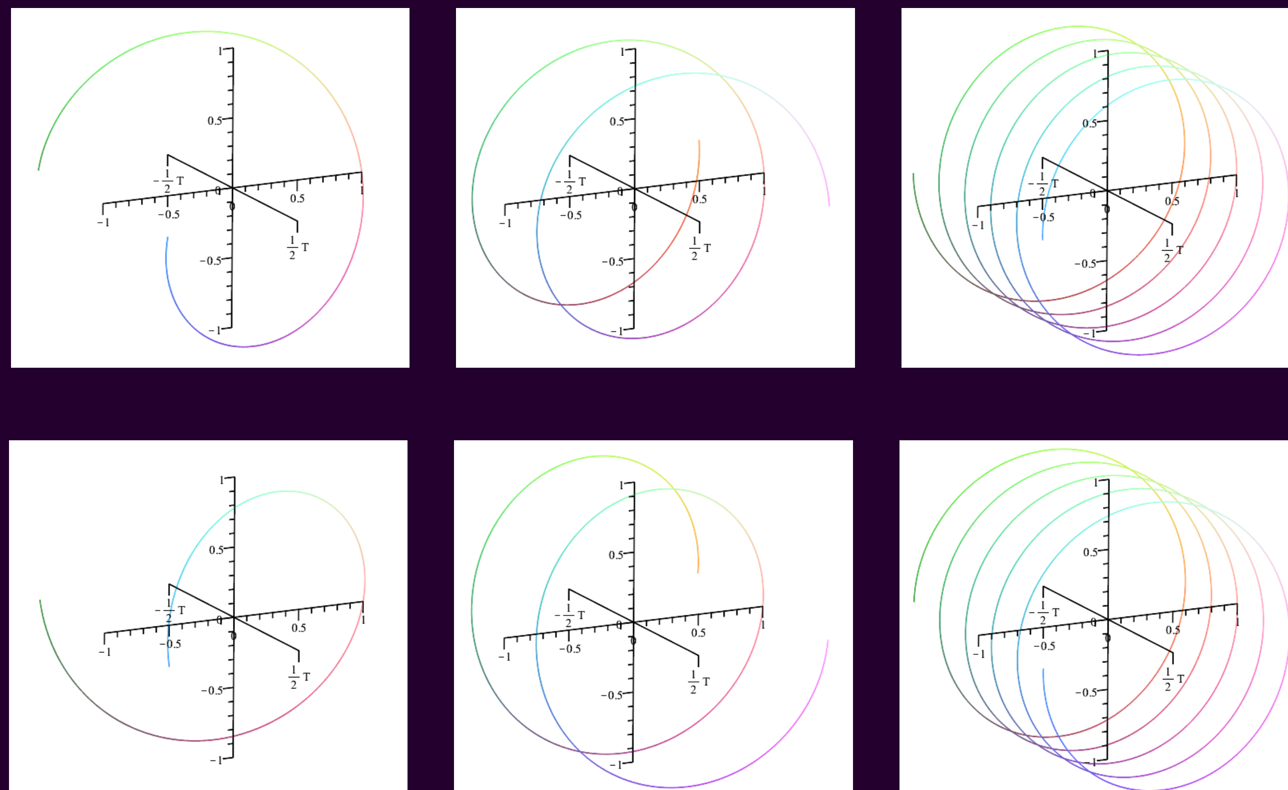
  - At the very least, the span contains all periodic functions with period T (including constant functions) with a finite number of discontinuities on any period

Orthogonal complex exponentials

- What do these functions look like?



$u_0(t) = 1$



$u_{\pm1}(t) = e^{\pm j\frac{2\pi}{T}t}$ $u_{\pm2}(t) = e^{\pm j\frac{4\pi}{T}t}$ $u_{\pm5}(t) = e^{\pm j\frac{10\pi}{T}t}$

A Fourier approximation

- Given a function $f(t)$, we'd like to find the coefficients

$$f(t) = \cdots + a_{-2} e^{-j\frac{4\pi}{T}t} + a_{-1} e^{-j\frac{2\pi}{T}t} + a_0 + a_1 e^{j\frac{2\pi}{T}t} + a_2 e^{j\frac{4\pi}{T}t} + \cdots$$

$$a_k = \frac{\langle u_k, f \rangle}{\langle u_k, u_k \rangle} = \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(e^{jk\frac{2\pi}{T}t}\right)^* f(t)\,\mathrm{d}t}{T} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-jk\frac{2\pi}{T}t} f(t)\,\mathrm{d}t$$

$$\int_a^b e^{st} t^n \,\mathrm{d}t = \frac{1}{s} e^{st} t^n \Big|_a^b - \frac{n}{s}\int_a^b e^{st} t^{n-1}\,\mathrm{d}t$$

A Fourier approximation

- If $f(t)$ is real periodic function, then $a_{-k} = a_k^*$
  - Also, $f(t)$ is real periodic function, then $a_{-k} = a_k^*$

$$a_k e^{jk\frac{2\pi}{T}t} + a_k^* e^{-jk\frac{2\pi}{T}t} = a_k e^{jk\frac{2\pi}{T}t} + \left(a_k e^{jk\frac{2\pi}{T}t} \right)^*$$

  - Without going through the math, this equals

$$2|a_k|\cos\left(k\frac{2\pi}{T}t + \arg(a_k) \right)$$

A Fourier approximation

- Consider the periodic extension of the polynomial t on [–1, 1] which creates a saw-tooth function

$$a_0 = 0$$

$$a_1 = -\frac{j}{\pi}$$

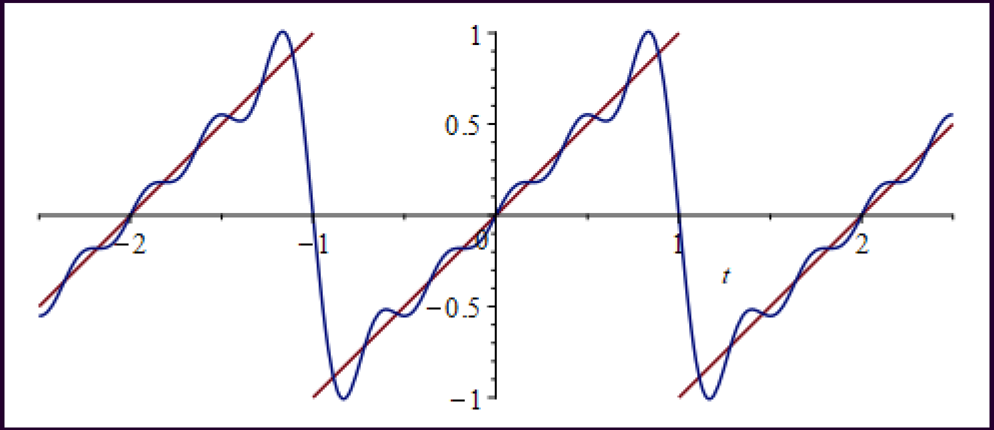
$$a_2 = \frac{j}{2\pi}$$

$$a_3 = -\frac{j}{3\pi}$$

$$a_4 = \frac{j}{4\pi}$$

$$a_5 = \frac{j}{5\pi}$$

$$\frac{2}{\pi}\cos(\pi t - 90°) + \frac{1}{\pi}\cos(2\pi t + 90°) + \frac{2}{3\pi}\cos(3\pi t - 90°) + \frac{1}{2\pi}\cos(4\pi t + 90°) + \frac{2}{5\pi}\cos(5\pi t + 90°)$$

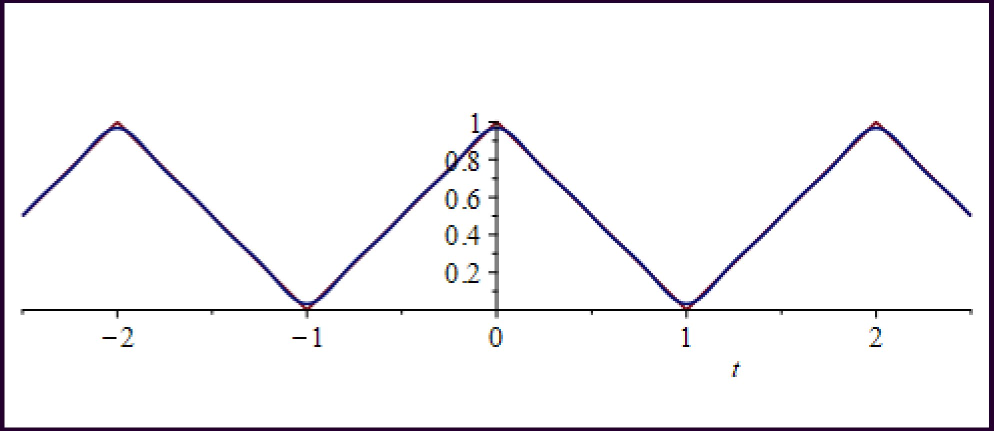


A Fourier approximation

- Consider the periodic extension of the function $1 - |t|$ on [–1, 1] which creates a periodic tent function

$$a_0 = \frac{1}{2}$$
$$a_1 = \frac{2}{\pi^2}$$
$$a_2 = 0$$
$$a_3 = \frac{2}{9\pi^2}$$
$$a_4 = 0$$
$$a_5 = \frac{2}{25\pi^2}$$

$$\frac{1}{2} + \frac{4}{\pi^2}\cos(\pi t) + \frac{4}{9\pi^2}\cos(3\pi t) + \frac{4}{25\pi^2}\cos(5\pi t)$$



A Fourier approximation

- Consider the periodic extension of (t – 1) (t + 1) (t – ½) on [–1, 1] which creates a non-symmetric wave

$$a_0 = \frac{1}{3}$$

$$a_1 = \frac{1}{\pi^2} + \frac{6}{\pi^3} j$$

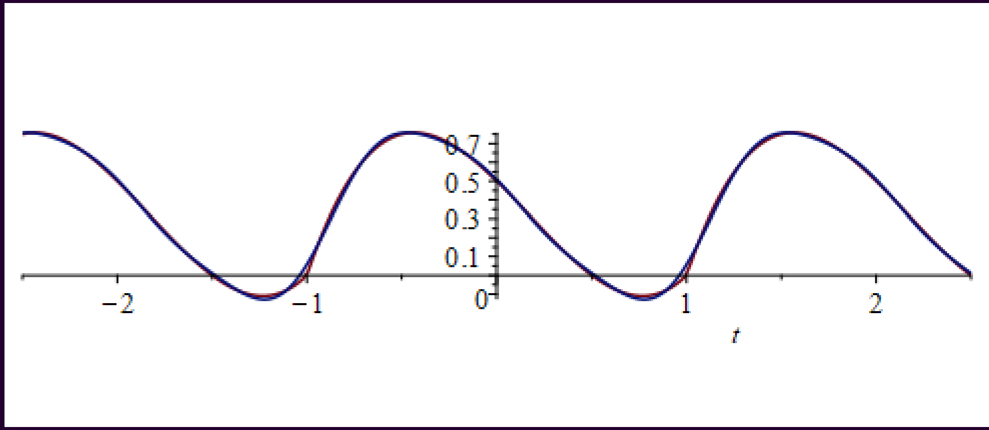
$$a_2 = -\frac{1}{4\pi^2} - \frac{6}{12\pi^3} j$$

$$a_3 = \frac{1}{9\pi^2} + \frac{6}{27\pi^3} j$$

$$a_4 = -\frac{1}{16\pi^2} - \frac{6}{32\pi^3} j$$

$$a_5 = \frac{1}{25\pi^2} + \frac{6}{125\pi^3} j$$

$$0.33 + 0.44\cos(\pi t + 62°) + 0.070\cos(2\pi t - 136°) + 0.027\cos(3\pi t + 32°) + 0.014\cos(4\pi t - 154°) + 0.0087\cos(5\pi t + 21°)$$



Maple code

- Here is the Maple code:

```
> restart;
> T := 2;
> p := t -> (t - 1)*(t + 1)*(t - 1/2);
> N := 5;
> interface( imaginaryunit = 'j' );
> for k from -N to N do
  a[k] := int( exp( -j*2*Pi/T*k*t )*p(t), t = -T/2..T/2 )/T;
end do;
> plots[display](
    plot( p(t), t = -0.5*T..0.5*T ),
    plot( add( a[k]*exp( j*2*Pi/T*k*t ), k = -N..N ),
                t = -1.25*T..1.25*T )
);
> a[0] + add( 2*abs(a[k])*cos(2*Pi/T*t + argument(a[k])),
              k = 1..N );
```

Other Fourier series

- There are other collections of orthogonal functions that use trigonometric functions

$$1, \cos\left(\frac{2\pi}{T}t\right), \cos\left(2\frac{2\pi}{T}t\right), \cos\left(3\frac{2\pi}{T}t\right), \ldots$$

$$\sin\left(\frac{2\pi}{T}t\right), \sin\left(2\frac{2\pi}{T}t\right), \sin\left(3\frac{2\pi}{T}t\right), \ldots$$

  - This is easier to visualize
  - This is more frustrating to calculate with twice as many integrals and integrals involving trigonometric functions
  - Also, a phase shifted cosine is easier to understand than a sum of two trigonometric functions

$$1.97\cos(\pi t + 40^\circ) \text{ versus } 1.509\cos(\pi t) - 1.266\sin(\pi t)$$

Summary

- Following this topic, you now
  - Understand the inner product of two functions
  - Know that two functions can be *orthogonal*
  - Know how to project one function onto another
  - Are aware that the sum of projections onto orthogonal functions gives the best approximation of the given function in terms of those orthogonal functions
  - Know about the complex exponential functions $u_n(t) = e^{jn\frac{2\pi}{T}t}$
  - Know these are orthogonal and form a basis for functions of period T
  - Have seen examples of approximating periodic functions with a finite number of these complex exponential functions

References

[1] https://en.wikipedia.org/wiki/Fourier_series#Complex-valued_functions

[2] Maplesoft: https://www.Maplesoft.com/

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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